The violation of uniqueness theorems for power series solutions by applying the Poincaré–Perron theorem

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Abstract

We show that uniqueness theorems are not available any more since we apply the Poincaré–Perron theorem into linear ordinary differential equations (ODEs). We verify that its theorem is only applicable when its power series solutions are absolutely convergent.

Keywords: Uniqueness theorems; Poincaré–Perron theorem; Heun functions; Three term recurrence relation
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1. Heun function and its domain of convergence

In the classical viewpoint and specialized flavour, power series solutions (the Maclaurin series) have been obtained by putting power series with unknown coefficients into linear ODEs. The recurrence relation of coefficients starts to arise, and there can be between 2-term and infinity-term in the recursive relation.

The canonical form of the general Heun's differential equation is taken as [18]

$$\frac{d^2y}{dx^2} + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a}\right)\frac{dy}{dx} + \frac{\alpha\beta x - q}{x(x-1)(x-a)}y = 0 \tag{1.1}$$

This equation is of Fuchsian type with regular singularities at $x = 0, 1, a, \infty$ with the condition $\epsilon = \alpha + \beta - \gamma - \delta + 1$ to ensure regularity of the point at ∞. The parameters play different roles: $a \neq 0$ is the singularity parameter, $\alpha, \beta, \gamma, \delta, \epsilon$ are exponent parameters, q is the accessory parameter. Also, α and β are identical to each other. The total number of free parameters is six. It has four regular singular points which are 0, 1, a and ∞ with exponents $\{0, 1-\gamma\}$, $\{0, 1-\delta\}$, $\{0, 1-\epsilon\}$ and $\{\alpha, \beta\}$.

The 2-term recursive relation starts to appear as we all know since the Maclaurin series are applied to hypergeometric-type equations (such as Bessel, Legendre, associated Legendre, Kummer, Airy equations, and etc).

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In contrast, a 3-term recurrence relation starts to appear since we put a function $y(x) = \sum_{n=0}^{\infty} c_n x^n$ into (1.1). Heun's equation has 4 types of confluent forms such as confluent Heun, triconfluent Heun, biconfluent Heun and double confluent Heun equations, and these 4 equations are obtained by combining two or more regular singularities to take form an irregular singularity in (1.1). This converting process to 4 confluent forms is similar to deriving of confluent hypergeometric equation from the hypergeometric equation. Also, coefficients entering power series solutions represented by confluent Heun, biconfluent and double confluent Heun equations satisfy the 3-term recursion relations, but strangely triconfluent Heun equation is composed of the 4-term recursive relation between successive coefficients in the power series. The Mathieu, Lame, spheroidal wave, hypergeometric-type equations and etc are just particular cases of the Heun's equation, referred as the 21^{th} century successor of hypergeometric equation [16, 30].

Heun's equation appears in mathematical physics problems, in addition to that, it starts to arise in economic and financial problems (SABR model) recently [15, 20]. For instance, the Heun functions come out in atomic and nuclear physics (the hydrogen-molecule ion) [37], in the Schrödinger equation with doubly anharmonic potential [27] (its solution is the confluent forms of Heun functions), in the Stark effect [13], in gravitational waves and black holes such as perturbations of the Kerr metric [1, 2, 3, 22, 36], in solid state physics (crystalline materials) [29], in Collogero–Moser–Sutherland systems [34], water molecule, graphene electrons[17, 28], quantum Rabi model [39], biophysics [31], etc [4, 32, 33].

Even though the Heun equation is represented in many scientific areas, its closed forms are unknown until now because the recurrence relation in its series solution consists of the 3-term. It seems that its general solution cannot be reduced to a 2-term recurrence relation by changing independent variables and coefficients [19]. Until recently, we have constructed many physics solutions by utilizing hypergeometric-type functions having the 2-term recursive relation between consecutive coefficients in their series solutions traditionally. But since the Heun's equation starts to appear in tremendous physical phenomenons, we cannot use hypergeometric functions any more. Today, it seems we require at least a three or four term recurrence relation in power series solutions.

Since the solution of (1.1) is analytic, $y(x)$ can be represented by a power series of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+\lambda} \tag{1.2}$$

where λ is an indicial root. Plug (1.2) into (1.1):

$$c_{n+1} = A_n \; c_n + B_n \; c_{n-1} \qquad ; n \geq 1 \tag{1.3}$$

where

$$\begin{aligned} A_n &= \frac{(n+\lambda)(n-1+\gamma+\epsilon+\lambda+a(n-1+\gamma+\lambda+\delta))+q}{a(n+1+\lambda)(n+\gamma+\lambda)} \\ &= \frac{(n+\lambda)(n+\alpha+\beta-\delta+\lambda+a(n+\delta+\gamma-1+\lambda))+q}{a(n+1+\lambda)(n+\gamma+\lambda)} \end{aligned} \tag{1.4a}$$

$$B_n = -\frac{(n-1+\lambda)(n+\gamma+\delta+\epsilon-2+\lambda)+\alpha\beta}{a(n+1+\lambda)(n+\gamma+\lambda)} = -\frac{(n-1+\lambda+\alpha)(n-1+\lambda+\beta)}{a(n+1+\lambda)(n+\gamma+\lambda)} \tag{1.4b}$$

$$c_1 = A_0 \; c_0 \tag{1.4c}$$

We have two indicial roots which are $\lambda = 0$ and $1-\gamma$.

For the domain of the Heun function $y(x)$ about $x = 0$, let $n \rightarrow \infty$ in (1.4a) and (1.4b)

$$\lim_{n\to\infty} A_n = A = \frac{1+a}{a} \tag{1.5a}$$

$$\lim_{n\to\infty} B_n = B = -\frac{1}{a} \tag{1.5b}$$

The recursive relation of (1.5a) and (1.5b) is

$$c_{n+1} = Ac_n + Bc_{n-1} \qquad ; n \geq 1 \tag{1.6}$$

Assuming $c_0 = 1$ for simplicity and letting $c_1 = Ac_0$.In Sec.2 [7], the asymptotic series of (1.6) for an infinite series is given by

$$\begin{aligned}
\lim_{n\to\infty} y(x) &= \sum_{k=0}^{\infty} y_k(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(n+m)!}{n!\; m!} (Bx^2)^n (Ax)^m = \frac{1}{1-(Ax+Bx^2)} \\
&= \frac{1}{1-\left(\frac{1+a}{a}x - \frac{1}{a}x^2\right)}
\end{aligned} \tag{1.7}$$

where

$$\left|\frac{1+a}{a}x\right| + \left|-\frac{1}{a}x^2\right| < 1 \tag{1.8}$$

(1.7) is the asymptotic series of Heun equation around $x = 0$ for an infinite series, and it is the geometric series.

The sequence c_n combines into combinations of A and B terms in (1.6): This is done by letting A in the sequence c_n is the leading term in the power series $y(x) = \sum_{n=0}^{\infty} c_n x^n$; we observe the term of sequence c_n which includes zero term of $A's$ for a sub-power series $y_0(x)$, one term of $A's$ for the sub-power series $y_1(x)$, two terms of $A's$ for a $y_2(x)$, three terms of $A's$ for a $y_3(x)$, etc.

The coefficient a decides the range of an independent variable x as we see the domain of (1.7). For $a, x \in \mathbb{R}$, all possible ranges of a and x in (1.8) are given in Table 1.

Range of the coefficient a	Range of the independent variable x
As $a = 0$	no solution
As $a > 0$	$\lvert x\rvert \leq \frac{1}{2}(-1-a+\sqrt{a^2+6a+1})$
As $-1 < a < 0$	$a < x \leq -a$
As $a \leq -1$	$\lvert x\rvert < 1$

Table 1: Domain of convergence of a Heun function about $x = 0$

Actually, $|x| = \frac{1}{2}(-1-a+\sqrt{a^2+6a+1})$ as $a > 0$ and $x = -a$ as $-1 < a < 0$ do not satisfied with the radius of convergence (1.8). But, for special cases, we can have the asymptotic series expansion at the interval of convergence $|(1+a)x/a| + \left|-x^2/a\right| = 1$.

First, let's think about the following double series such as

$$\lim_{n\to\infty} y(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(n+m)!}{n!\; m!} x^n y^m \tag{1.9}$$

generally speaking, where $|x| + |y| < 1$.

We suggest that $x = -\xi$ and $y = 1 - \xi$ where $0 < \xi < 1$, and (1.9) is

$$\lim_{n\to\infty} y(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(n+m)!}{n!\; m!}(-\xi)^n (1-\xi)^m = \frac{1}{2\xi} \tag{1.10}$$

Second, put $x = -\xi$ and $y = \xi - 1$ where $0 < \xi < 1$ into (1.9)

$$\lim_{n\to\infty} y(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(n+m)!}{n!\; m!}(-\xi)^n (\xi-1)^m = \frac{1}{2} \tag{1.11}$$

As we see (1.10) and (1.11), we obtain approximated series solution even if $|x| + |y| = 1$. For the Heun case, put $\xi = x^2/a$ where $0 < x^2/a < 1$ into (1.10) and (1.11) after that, substitute $\xi = -(1+a)x/a$ where $0 < -(1+a)x/a < 1$ into the same two equations. Then, we obtain corrected domain of convergence including two proper variables such as $|x| = \frac{1}{2}(-1 - a + \sqrt{a^2 + 6a + 1})$ as $a > 0$ and $x = -a$ as $-1 < a < 0$.

For $a, x \in \mathbb{R}$, Fig. 1 represents a graph for the radius of convergence of Table 1 in the a-x plane; the shaded area represents the domain of convergence of the series for a Heun's equation around $x = 0$; it does not include dotted lines but solid lines, there is no such solution at the origin (the black colored point), and maximum modulus of x is the unity.

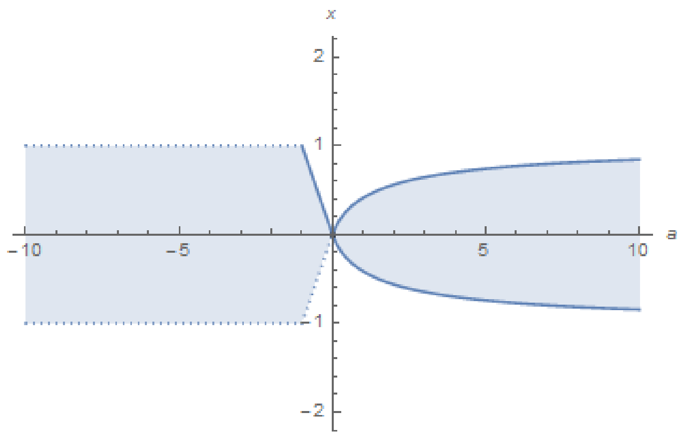

Figure 1: Domain of convergence of the series (1.7)

By rearranging coefficients A_n and B_n in each sequence c_n in (1.3) where $c_0 = 1$, the Maclaurin series solutions of the Heun equation are given by [6]

$$y^A(x) = x^\lambda \left(y_0^A(x) + y_1^A(x)\eta + \sum_{n=2}^{\infty} y_n^A(x)\eta^n \right) \tag{1.12}$$

where

$$
\begin{aligned}
y_0^A(x) &= \sum_{i_0=0}^{\infty} \frac{(\frac{\alpha}{2}+\frac{\lambda}{2})_{i_0}(\frac{\beta}{2}+\frac{\lambda}{2})_{i_0}}{(1+\frac{\lambda}{2})_{i_0}(\frac{1}{2}+\frac{\gamma}{2}+\frac{\lambda}{2})_{i_0}} z^{i_0} \\
y_1^A(x) &= \sum_{i_0=0}^{\infty} \frac{(i_0+\frac{\lambda}{2})\left(i_0+\Gamma_0^{(I)}\right)+Q}{(i_0+\frac{1}{2}+\frac{\lambda}{2})(i_0+\frac{\gamma}{2}+\frac{\lambda}{2})} \frac{(\frac{\alpha}{2}+\frac{\lambda}{2})_{i_0}(\frac{\beta}{2}+\frac{\lambda}{2})_{i_0}}{(1+\frac{\lambda}{2})_{i_0}(\frac{1}{2}+\frac{\gamma}{2}+\frac{\lambda}{2})_{i_0}} \\
&\quad \times \sum_{i_1=i_0}^{\infty} \frac{(\frac{1}{2}+\frac{\alpha}{2}+\frac{\lambda}{2})_{i_1}(\frac{1}{2}+\frac{\beta}{2}+\frac{\lambda}{2})_{i_1}(\frac{3}{2}+\frac{\lambda}{2})_{i_0}(1+\frac{\gamma}{2}+\frac{\lambda}{2})_{i_0}}{(\frac{1}{2}+\frac{\alpha}{2}+\frac{\lambda}{2})_{i_0}(\frac{1}{2}+\frac{\beta}{2}+\frac{\lambda}{2})_{i_0}(\frac{3}{2}+\frac{\lambda}{2})_{i_1}(1+\frac{\gamma}{2}+\frac{\lambda}{2})_{i_1}} z^{i_1} \\
y_n^A(x) &= \sum_{i_0=0}^{\infty} \frac{(i_0+\frac{\lambda}{2})\left(i_0+\Gamma_0^{(I)}\right)+Q}{(i_0+\frac{1}{2}+\frac{\lambda}{2})(i_0+\frac{\gamma}{2}+\frac{\lambda}{2})} \frac{(\frac{\alpha}{2}+\frac{\lambda}{2})_{i_0}(\frac{\beta}{2}+\frac{\lambda}{2})_{i_0}}{(1+\frac{\lambda}{2})_{i_0}(\frac{1}{2}+\frac{\gamma}{2}+\frac{\lambda}{2})_{i_0}} \\
&\quad \times \prod_{k=1}^{n-1} \left\{ \sum_{i_k=i_{k-1}}^{\infty} \frac{(i_k+\frac{k}{2}+\frac{\lambda}{2})\left(i_k+\Gamma_k^{(I)}\right)+Q}{(i_k+\frac{k}{2}+\frac{1}{2}+\frac{\lambda}{2})(i_k+\frac{k}{2}+\frac{\gamma}{2}+\frac{\lambda}{2})} \right. \\
&\quad \left. \times \frac{(\frac{k}{2}+\frac{\alpha}{2}+\frac{\lambda}{2})_{i_k}(\frac{k}{2}+\frac{\beta}{2}+\frac{\lambda}{2})_{i_k}(1+\frac{k}{2}+\frac{\lambda}{2})_{i_{k-1}}(\frac{1}{2}+\frac{k}{2}+\frac{\gamma}{2}+\frac{\lambda}{2})_{i_{k-1}}}{(\frac{k}{2}+\frac{\alpha}{2}+\frac{\lambda}{2})_{i_{k-1}}(\frac{k}{2}+\frac{\beta}{2}+\frac{\lambda}{2})_{i_{k-1}}(1+\frac{k}{2}+\frac{\lambda}{2})_{i_k}(\frac{1}{2}+\frac{k}{2}+\frac{\gamma}{2}+\frac{\lambda}{2})_{i_k}} \right\} \\
&\quad \times \sum_{i_n=i_{n-1}}^{\infty} \frac{(\frac{n}{2}+\frac{\alpha}{2}+\frac{\lambda}{2})_{i_n}(\frac{n}{2}+\frac{\beta}{2}+\frac{\lambda}{2})_{i_n}(1+\frac{n}{2}+\frac{\lambda}{2})_{i_{n-1}}(\frac{1}{2}+\frac{n}{2}+\frac{\gamma}{2}+\frac{\lambda}{2})_{i_{n-1}}}{(\frac{n}{2}+\frac{\alpha}{2}+\frac{\lambda}{2})_{i_{n-1}}(\frac{n}{2}+\frac{\beta}{2}+\frac{\lambda}{2})_{i_{n-1}}(1+\frac{n}{2}+\frac{\lambda}{2})_{i_n}(\frac{1}{2}+\frac{n}{2}+\frac{\gamma}{2}+\frac{\lambda}{2})_{i_n}} z^{i_n} \qquad (1.13)
\end{aligned}
$$

and

$$
\begin{cases}
\eta = \frac{(1+a)}{a} x \\
z = -\frac{1}{a} x^2 \\
\Gamma_0^{(I)} = \frac{1}{2(1+a)}(\alpha+\beta-\delta+\lambda+a(\delta+\gamma-1+\lambda)) \\
\Gamma_k^{(I)} = \frac{1}{2(1+a)}(\alpha+\beta-\delta+k+\lambda+a(\delta+\gamma-1+k+\lambda)) \\
Q = \frac{q}{4(1+a)}
\end{cases}
$$

The sequence c_n combines into combinations of A_n and B_n terms in (1.3): (1.12) and (1.13) are done by letting A_n in the sequence c_n is the leading term in a series (1.2); we observe the term of sequence c_n which includes zero term of $A_n's$ for a sub-power series $y_0^A(x)$, one term of $A_n's$ for the sub-power series $y_1^A(x)$, two terms of $A_n's$ for a $y_2^A(x)$, three terms of $A_n's$ for a $y_3^A(x)$, etc.

In chap 2. [8], similarly, by letting B_n in the sequence c_n is the leading term in a series (1.2), another solution is taken by

$$
y^B(x) = x^\lambda \left(y_0^B(x) + y_1^B(x) z + \sum_{n=2}^{\infty} y_n^B(x) z^n \right) \qquad (1.14)
$$

where

$$y_0^B(x) = \sum_{i_0=0}^{\infty} \frac{\left(\Delta_0^-\right)_{i_0} \left(\Delta_0^+\right)_{i_0}}{(1+\lambda)_{i_0}(\gamma+\lambda)_{i_0}} \eta^{i_0}$$

$$y_1^B(x) = \sum_{i_0=0}^{\infty} \frac{(i_0+\lambda+\alpha)(i_0+\lambda+\beta)}{(i_0+\lambda+2)(i_0+\lambda+1+\gamma)} \frac{\left(\Delta_0^-\right)_{i_0} \left(\Delta_0^+\right)_{i_0}}{(1+\lambda)_{i_0}(\gamma+\lambda)_{i_0}} \sum_{i_1=i_0}^{\infty} \frac{\left(\Delta_1^-\right)_{i_1} \left(\Delta_1^+\right)_{i_1} (3+\lambda)_{i_0}(2+\gamma+\lambda)_{i_0}}{\left(\Delta_1^-\right)_{i_0} \left(\Delta_1^+\right)_{i_0} (3+\lambda)_{i_1}(2+\gamma+\lambda)_{i_1}} \eta^{i_1}$$

$$\begin{aligned} y_n^B(x) = {} & \sum_{i_0=0}^{\infty} \frac{(i_0+\lambda+\alpha)(i_0+\lambda+\beta)}{(i_0+\lambda+2)(i_0+\lambda+1+\gamma)} \frac{\left(\Delta_0^-\right)_{i_0} \left(\Delta_0^+\right)_{i_0}}{(1+\lambda)_{i_0}(\gamma+\lambda)_{i_0}} \\ & \times \prod_{k=1}^{n-1} \left\{ \sum_{i_k=i_{k-1}}^{\infty} \frac{(i_k+2k+\lambda+\alpha)(i_k+2k+\lambda+\beta)}{(i_k+2(k+1)+\lambda)(i_k+2k+1+\gamma+\lambda)} \frac{\left(\Delta_k^-\right)_{i_k} \left(\Delta_k^+\right)_{i_k} (2k+1+\lambda)_{i_{k-1}}(2k+\gamma+\lambda)_{i_{k-1}}}{\left(\Delta_k^-\right)_{i_{k-1}} \left(\Delta_k^+\right)_{i_{k-1}} (2k+1+\lambda)_{i_k}(2k+\gamma+\lambda)_{i_k}} \right\} \\ & \times \sum_{i_n=i_{n-1}}^{\infty} \frac{(\Delta_n^-)_{i_n} (\Delta_n^+)_{i_n} (2n+1+\lambda)_{i_{n-1}}(2n+\gamma+\lambda)_{i_{n-1}}}{(\Delta_n^-)_{i_{n-1}} (\Delta_n^+)_{i_{n-1}} (2n+1+\lambda)_{i_n}(2n+\gamma+\lambda)_{i_n}} \eta^{i_n} \end{aligned} \tag{1.15}$$

and

$$\begin{cases} \eta = \frac{(1+a)}{a} x \\ z = -\frac{1}{a} x^2 \\ \Delta_0^{\pm} = \frac{\varphi+2(1+a)\lambda \pm \sqrt{\varphi^2-4(1+a)q}}{2(1+a)} \\ \Delta_1^{\pm} = \frac{\varphi+2(1+a)(\lambda+2) \pm \sqrt{\varphi^2-4(1+a)q}}{2(1+a)} \\ \Delta_k^{\pm} = \frac{\varphi+2(1+a)(\lambda+2k) \pm \sqrt{\varphi^2-4(1+a)q}}{2(1+a)} \\ \Delta_n^{\pm} = \frac{\varphi+2(1+a)(\lambda+2n) \pm \sqrt{\varphi^2-4(1+a)q}}{2(1+a)} \end{cases}$$

$y^A(x)$ and $y^B(x)$ are equivalent to each other analytically as long as the domain $\left|\frac{1+a}{a}x\right| + \left|-\frac{1}{a}x^2\right| < 1$ is held.

2. Poincaré–Perron theorem and its applications to Heun's equation

According to the Poincaré–Perron theory [5, 25, 23, 24], the characteristic polynomial of (1.6) is given by

$$t^2 - At - B = 0 \tag{2.1}$$

The spectral numbers λ_1 and λ_2 in (2.1) have two different moduli such as

$$\lambda_1 = \frac{1+a-|1-a|}{2a} \qquad \lambda_2 = \frac{1+a+|1-a|}{2a} \tag{2.2}$$

More explanations are explained in Appendix B of part A [27], Wimp (1984) [38], Kristensson (2010) [21] or Erdélyi (1955) [14].

The Poincaré–Perron theory for the 3-term recurrence relation states that,

(i) if $|\lambda_i| < |\lambda_j|$, then $\lim_{n\to\infty} |c_{n+1}/c_n| = |\lambda_j|$, so that the radius of convergence for a 3-term recursion relation (2.1) is $|\lambda_j|^{-1}$ where $i, j = \{1, 2\}$.

(ii) if $|\lambda_i| = |\lambda_j|$ and $\lambda_i \neq \lambda_j$, then $\lim_{n\to\infty} |c_{n+1}/c_n|$ does not exist; if $\lambda_i = \lambda_j$, $\lim_{n\to\infty} |c_{n+1}/c_n|$ is convergent.

For $a, x \in \mathbb{R}$, the radius of convergence for (2.2) using (i) and (ii) is given in Table 2

Range of the coefficient a	Range of the independent variable x
As $a = 0$	no solution
As $|a| \geq 1$	$|x| < 1$
As $-1 < a < 0$	$a < x \leq -a$
As $0 < a < 1$	$-a \leq x < a$

Table 2: Domain of convergence of a Heun function about $x = 0$ using Poincaré–Perron theory

Actually, there is no such solution at $a = -1$ in Table 2 because of the statement (ii). But we are able to have suitable domain of a Heun function as long as the Poincaré–Perron theory give us a series as conditionally convergent because of Thm.1. To verify this phenomenon, put $a = -1$ into (1.5a) and (1.5b), and substitute the new (1.5a) and (1.5b) in (1.6)

$$c_{n+1} = c_{n-1} \qquad ; n \geq 1 \tag{2.3}$$

where $c_0 = 1$ and $c_1 = 0$. Put (2.3) into a series $\sum_{n=0}^{\infty} c_n x^n$

$$\lim_{n\to\infty} y(x) = \sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2} \tag{2.4}$$

Therefore, we have the proper domain $|x| < 1$ as $a = -1$ for the conditional series solution.

Similarly, according to the the statement (i), we cannot obtain the conditional solution at $x = -a$ where $-1 < a < 0$, $0 < a < 1$. But, for the particular case, we can have corrected series solution at the boundary of the domain even if $|x| = |\lambda_j|^{-1}$. Put (1.5a) and (1.5b) in (1.6)

$$c_{n+1} = \frac{1+a}{a} c_n - \frac{1}{a} c_{n-1} \qquad ; n \geq 1 \tag{2.5}$$

where $c_1 = \frac{1+a}{a}$ and $c_0 = 1$. And insert (2.5) into $\sum_{n=0}^{\infty} c_n (-a)^n$

$$\lim_{n\to\infty} y(x) = \sum_{n=0}^{\infty} a^{2n} = \frac{1}{1-a^2} \tag{2.6}$$

It is converge at $-1 < a < 0$, $0 < a < 1$.

The proper domain of convergence in the real axis, given by Table 2, is shown shaded in Fig. 2; it does not include dotted lines but solid lines, there is no such solution at the origin, and maximum modulus of x is the unity.

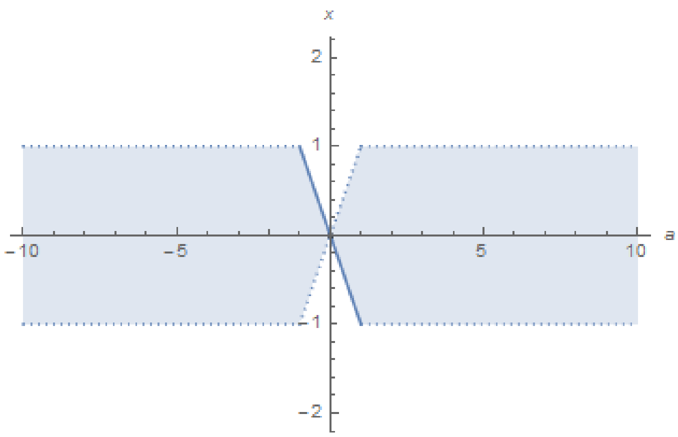

Figure 2: Domain of convergence of the series by applying Poincaré-Perron theorem

Unlike series solutions (1.12)–(1.15), taken by rearranging coefficients A_n and B_n in each sequence c_n in (1.3); a power series solutions on the domain of convergence which is obtained by Poincaré–Perron theory, is just left as a general solution of Heun equation as a solution of recurrences.

$$\begin{aligned} y^P(x) &= \sum_{n=0}^{\infty} c_n x^{n+\lambda} \\ &= x^{\lambda}\Big(1 + A_0 x + (B_1 + A_{0,1})\, x^2 + (A_0 B_2 + A_2 B_1 + A_{0,1,2})\, x^3 \\ &\quad + (B_{1,3} + A_{0,1} B_3 + A_{0,3} B_2 + A_{2,3} B_1 + A_{0,1,2,3})\, x^4 \\ &\quad + (A_0 B_{2,4} + A_2 B_{1,4} + A_4 B_{1,3} + A_{0,1,2} B_4 + A_{0,1,4} B_3 + A_{0,3,4} B_2 \\ &\quad + A_{2,3,4} B_1 + A_{0,1,2,3,4})\, x^5 + \cdots\Big) \end{aligned} \tag{2.7}$$

In (2.7), the definition of $B_{i,j,k,l}$ refer to $B_i B_j B_k B_l$. Also, $A_{i,j,k,l}$ refer to $A_i A_j A_k A_l$.

Tables 1 and 2 tell us that both intervals of convergence of Heun function at $x = 0$ are not equal to each other analytically. Because Table 1 is build by rearranging coefficients A and B in each sequence c_n in (1.6). In contrast, Table 2 is taken by observing the ratio of sequence c_{n+1} to c_n at the limit $n \rightarrow \infty$ in (1.6).

Fig.3 represents two different shaded areas of convergence in Figs. 1 and 2: (1) There are no analytic solutions at $a = 0$ for both domains of convergence, (3) in the bright shaded area where $a > 0$, the domain of convergence of the Heun series around $x = 0$, obtained by rearranging coefficients A and B terms in each sequence c_n in (1.6), is not available; it only provides the domain of convergence obtained by the Poincaré–Perron theory, (4) the dark shaded region where $a > 0$ represents the radius of convergence in Tables 1, (5) in the dark shaded area where $a < 0$, two different domains of convergence are equivalent to each other.

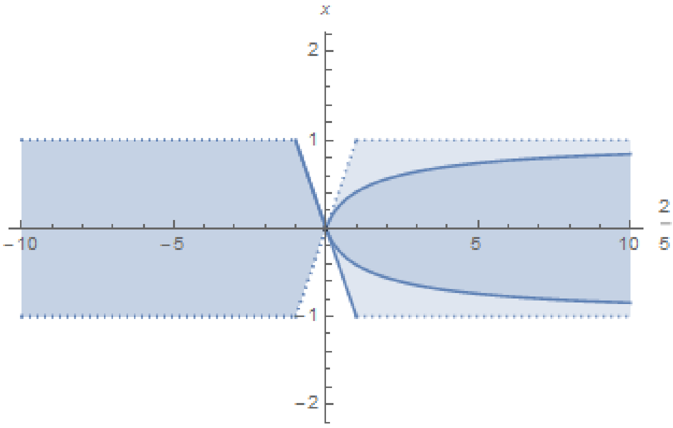

Figure 3: Two different domains of Tables 1 and 2

Theorem 1. *We can not use Poincaré–Perron theorem to obtain the radius of convergence for a power series solution. And a series solution for an infinite series, obtained by applying Poincaré–Perron theorem, is not absolute convergent but only conditionally convergent.*

It has been believed that the Poincaré–Perron theory provides us the domain of convergence for a power series solution of Heun equation. If this is true, a series solution converges absolutely whether we rearrange of its terms for the series solution or not. However, At any points of (a, x) in the bright shaded area where $a > 0$ on Fig.3, an approximated geometric series solution of (1.7), constructed by rearranging of A and B terms in (1.6), is not convergent any more. Also, $y^A(x)$ and $y^B(x)$ will be the divergent infinite series in (1.12)–(1.15).

By putting $\sum_{n=0}^{\infty} c_n x^n$ into (1.6) with $c_0 = 1$, the series solution of the 3-term recurrence relation is taken by

$$\sum_{n=0}^{\infty} c_n x^n = 1 + Ax + \left(A^2 + B\right) x^2 + \left(A^3 + 2AB\right) x^3 + \left(A^4 + 3A^2B + B^2\right) x^4 + \left(A^5 + 4A^3B + 3AB^2\right) x^5 + \cdots \tag{2.8}$$

The absolutely convergent series solution is decided uniquely when $\sum_{n=0}^{\infty} |c_n||x|^n$ is convergent. According to the Cauchy ratio test, if the condition $\lim_{n\rightarrow\infty} \left|\frac{c_{n+1}}{c_n}\right| |x| < 1$ is satisfied, a series solution is absolute convergent. And the Poincaré–Perron theorem tell us that $\left|\frac{c_{n+1}}{c_n}\right|$ as $n \rightarrow \infty$ is equivalent to one of roots of the characteristic polynomial in (2.1). This approach gives us that a series solution is not absolutely convergent but conditional convergent. Taking all absolute values inside parentheses of (2.8) to make an absolutely convergent series solution, we obtain a proper radius of convergence for a solution.

$$\sum_{n=0}^{\infty} |c_n||x|^n = 1 + |A||x| + \left(\left|A^2\right| + |B|\right) |x|^2 + \left(\left|A^3\right| + |2AB|\right) |x|^3 + \left(\left|A^4\right| + \left|3A^2B\right| + \left|B^2\right|\right) |x|^4 + \left(\left|A^5\right| + \left|4A^3B\right| + \left|3AB^2\right|\right) |x|^5 + \cdots \tag{2.9}$$

If we take moduli of A and B in (2.1), its domain of convergence is equivalent to Table 1 except the case of $a = -1$. More explicit proof is explained in Sec.2 [7].

3. Uniqueness theorem and Poincaré–Perron theorem

According to Fig.3, we have

$$D_a = \text{Domain of absolute convergence } \subseteq D_c = \text{Domain of conditional convergence} \quad (3.1)$$

As we all know, (3.1) is not only available for the 3-term recurrence relation of linear ODEs but also multi-term cases.

Uniqueness theorem for Poisson's equation tells us that its equation has an unique solution for a given boundary conditions: If Ω is a boundary domain in $\mathbb{R}^n$, a function $u : \Omega \to \mathbb{R}$ such that $u \in C^2(\Omega) \cap C(\bar{\Omega})$, and $\nabla^2 u = \rho$ in Ω and either $u = f$ or $\partial u/\partial n = g$ on $\partial\Omega$. where ρ, f and g are given functions, then u is unique (at least to within an additive constant). [26] Also, its theorem is available in the Klein-Gordon equation; the solution for the scalar field $\Phi(r)$ in Ω bounded by $\partial\Omega$ is unique if either Dirichlet or Neumann boundary conditions are specified on $\partial\Omega$.

In general, Heun-type functions require the uniqueness theorem in mathematical physics: For instance, the Heun equation is derived from the Klein-Gordon equation for $D = 4$ Kerr-de Sitter metric. [10, 33, 35]

Suppose that a series solution $y^P(x)$ of the Heun function at $x = 0$ is unique and its domain of convergence D_c is obtained by the Poincaré–Perron theory. Then, a series $y^A(x)$ $(=y^B(x))$ should be equivalent or proportional to a solution $y^P(x)$ with a constant value. However, at the region $D_c - D_a$ on Fig.3 (in the bright shaded area where $a > 0$), a series $y^A(x)$ is divergent; it means that $y^P(x)$ and $y^A(x)$ are independent to each other. Therefore, the uniqueness theorem is broken and is not available any more by applying the Poincaré–Perron theorem.

Now, let assume that $y^A(x)$ is unique in a boundary domain D_a, then a function $y^P(x)$ is equivalent (or proportional) to a series $y^A(x)$ because of $D_a \subseteq D_c$: It means the uniqueness theorem is available in this situation. Because of this, we obtain the following theorem such that

Theorem 2. *For the multi-term recurrence relation of a homogeneous linear ODE (more than the 3-term), if a power series solution is conditionally convergent, then the uniqueness theorem is not available any more. Its theorem is only valid as a series is absolute convergent.*

For the Heun case, a domain of conditional convergence is constructed by utilizing the Poincaré–Perron theorem. In contrast, a domain of absolute convergence is build by by rearranging coefficients A and B in each sequence c_n in (1.6). We can extend this idea to the case of a multi-term recursion relation in a series solution, and the radius of convergence for the multi-term case and its asymptotic series are taken on pp. 69 in chap 3. [9]

For a fixed $k \in \mathbb{N}$, the $(k+1)$-term recurrence relation of a homogeneous linear ordinary differential equation about $x = 0$ with constant coefficients is

$$c_{n+1} = \alpha_1 c_n + \alpha_2 c_{n-1} + \alpha_3 c_{n-2} + \cdots + \alpha_k c_{n-k+1} \quad (3.2)$$

assuming $\alpha_l < \infty$ where $l \in \mathbb{N}$. And approximated series solution of $(k+1)$-term recurrence relation is taken by

$$\lim_{n\to\infty} y(x) = \sum_{n=0}^{\infty} c_n x^n = \frac{1}{1 - \sum_{m=1}^{k} \alpha_m x^m} \quad (3.3)$$

and its domain of absolute convergence is written as

$$D_a = \sum_{m=1}^{k} |\alpha_m x^m| < 1 \tag{3.4}$$

(3.4) is also required for the uniqueness theory, and its series solution obtained by rearranging α_l terms is dependent to a series left as a general solution of a linear ODE as a solution of recurrences.

If $D_a = D_c$, uniqueness theory is always permitted, for example, a series solution of the confluent Heun equation (the non-symmetrical canonical form) about $x = 0$ where $|x| < 1$ [11, 12, 27] is not only conditionally convergent but also absolute convergent. We obtain the same domain ($|x| < 1$) whether we use the Poincaré–Perron theorem or rearranging coefficients A and B terms.

Generally speaking, by observing linear ODEs having more than the 3-term, we notice that a power series solution can have two different domains which are conditionally convergent and absolute one at the same time. But we have to choose a series solution having an absolute convergence for the uniqueness theorem. Strangely, a series solution for the 2-term has only a single domain which is conditionally convergent and/or absolute one.

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